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Exact Solvability of Superintegrable Systems

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Abstract

It is shown that all four superintegrable quantum systems on the Euclidean plane possess the same underlying hidden algebra $sl(3)$. The gauge-rotated Hamiltonians, as well as their integrals of motion, once rewritten in appropriate coordinates, preserve a flag of polynomials. This flag corresponds to highest-weight finite-dimensional representations of the $sl(3)$-algebra, realized by first order differential operators.

1 Introduction

The purpose of this Letter is to establish a relation between two different concepts in quantum mechanics: *"superintegrability"* and *"exact solvability"*. More specifically we relate these two concepts in nonrelativistic quantum mechanics in two dimensional Euclidean space E_2.

The notion of integrability in quantum mechanics [1] comes naturally as a generalization of a similar notion in classical mechanics (see, for instance, [2]). A quantum mechanical system in E_n described by the stationary Schroedinger equation

$$H\Psi = E\Psi \ , \ H = -\frac{1}{2}\triangle + V(x_1,,\dots,x_n) \ , \tag{1}$$

is completely integrable if there exists a set of $(n-1)$ algebraically independent linear operators $X_a, a = 1,2,\dots,n-1$ commuting with the Hamiltonian and among each other

$$[H, X_a] = 0 \ , \ [X_a, X_b] = 0 \ . \tag{2}$$

The system is *"superintegrable"* if there exist k additional operators, $Y_b, b = 1,\dots,k$, where $0 < k \le (n-1)$, commuting with the Hamiltonian. It is "maximally superintegrable" if $k = n-1$.

The operators X_a, Y_b are usually assumed to be polynomials in the momenta $\{p_1,\dots,p_n\}$ with coordinate dependent coefficients. A systematic search for superintegrable systems in E_2 and E_3 was conducted some time ago [3, 4, 5]. A restriction was imposed, namely that the operators X_a and Y_b should be second order polynomials in momenta. It turned out that the existence of such commuting operators leads to the separation of variables in the Schroedinger equation. Superintegrable systems are actually *"super-separable"*: they allow the separation of variables in at least two coordinate systems.

A large body of literature exists on superintegrable systems in E_n [3] - [17]. In particular, it has been shown recently [17] that the superintegrable systems in E_2 are characterized by the existence of at least two generalized Lie symmetries.

Quantum mechanical problems which can be called *exactly solvable* are defined quite differently. They are characterized by the fact that one can indicate explicitly an infinite flag of functional linear spaces, which is preserved by the Hamiltonian [18]. We recall that a *flag* is formed by an infinite set of functional linear spaces which can be ordered in such a way that each of them contains the previous one as a subspace. One important particular

example of these flags is given by finite-dimensional representation spaces of semi-simple Lie algebras of first order differential operators. In this case the Hamiltonian is an element of the universal enveloping algebra of a Lie algebra.

In order to clarify the situation let us consider as an example the case of one dimensional (quasi)-exactly solvable problems[19]. Due to Sophus Lie it is known that the only Lie algebra of first order differential operators which acts on the real line and possesses finite-dimensional representations is the $sl(2,\mathbb{R})$-algebra (for a discussion see, for example, [21, 22]), realized as

$$J_n^+ = x^2 d_x - nx \quad , \quad J_n^0 = x d_x - \frac{n}{2} \quad , \quad J_n^- = d_x \ . \tag{3}$$

For integer n the generators (3) possess a common invariant subspace $\mathcal{P}_n = \langle x^k | 0 \leq k \leq n \rangle$, which is the linear space of polynomials. It is evident that the spaces $\mathcal{P}_n$ as functions of the parameter n form a flag. This flag is preserved by any element of the universal enveloping algebra of the $sl(2,\mathbb{R})$ parabolic subalgebra J_n^0, J_n^- for any n. Therefore, an element of this enveloping algebra can be viewed as a Hamiltonian, which defines an *exactly solvable* system. In a similar manner one can introduce the notion of a quasi-exactly solvable problem for which the Hamiltonian possesses the invariant subspace $\mathcal{P}_n$. It can be proven [23] that a necessary and sufficient condition for a one dimensional Hamiltonian to be quasi-exactly solvable is that it belongs to the universal enveloping algebra of the $sl(2,\mathbb{R})$-algebra taken in realization (3).

For two-dimensional problems there exist four candidates for underlying hidden Lie algebra [20, 21, 22, 18]: $sl(3,\mathbb{R})$, $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$, $o(3,1)$, a parametric family $gl(2,\mathbb{R}) \ltimes \mathbf{R}^{r+1}$ and some of their subalgebras. In particular, the algebra $gl(3,\mathbb{R}) \supset sl(3,\mathbb{R})$, realized as

$$\begin{gathered}
J_1 \ = \ \partial_t \ , \ J_2 \ = \ \partial_u \ , \\
J_3 \ = \ t\partial_t \ , \ J_4 \ = \ u\partial_u \ , \ J_5 \ = \ u\partial_t \ , \ J_6 \ = \ t\partial_u \ , \\
J_7 \ = \ t^2\partial_t \ + \ tu\partial_u \ - \ nt \ , \ J_8 \ = \ tu\partial_t \ + \ u^2\partial_u \ - \ nu \ , \\
X \ = \ n \ ,
\end{gathered} \tag{4}$$

will be used in this article as well as the maximal parabolic subalgebra of $sl(3,\mathbb{R})$ formed by the generators $\{J_{1,\ldots,6}\}$.

For integer n the generators (4) possess a common invariant subspace $\mathcal{P}_n^{(2)} = \langle t^k u^m | 0 \leq k+m \leq n \rangle$, which is the linear space of polynomials. Similarly to the case of $sl(2,\mathbb{R})$ the spaces $\mathcal{P}_n^{(2)}$ as functions of the parameter n

form a flag. This flag is preserved by any element of the universal enveloping algebra of the parabolic subalgebra $\{J_{1,\ldots,6}\}$ [18]. Therefore, an element of this enveloping algebra viewed as a Hamiltonian defines an *exactly solvable* system with the hidden algebra $sl(3,\mathbb{R})$.

Both superintegrable and exactly solvable systems have numerous applications in physics. Among the simplest superintegrable ones are the Coulomb system and the harmonic oscillator in spaces of any dimension. The celebrated many-body Calogero model is superintegrable as well as the Hartmann potential of quantum chemistry [10, 9].

2 Hidden Algebra of Superintegrable Hamiltonians

There exist precisely four quantum (and also classical) Hamiltonians defined on E_2, characterized by two integrals of motion, $[H, X_{1,2}] = 0$, such that $X_{1,2}$ are quadratic in the momenta [3, 4, 17]. Thus, they are maximally superintegrable and for their classical counterparts all trajectories are closed. It was shown that these four Hamiltonians exhaust the list of two-dimensional Hamiltonians characterized by two integrals of motion in the form of second order differential operators. They admit the separation of variables in two (or even more) different coordinate systems. We will show that all of them possess a hidden $sl(3,\mathbb{R})$ algebra. In particular, this implies that there exists a coordinate system where the Hamiltonian, as well as the integrals of motion, after a similarity transformation (gauge rotation) can be rewritten in terms of the generators of the maximal parabolic subalgebra of $sl(3,\mathbb{R})$. Furthermore, two of these Hamiltonians have a striking feature. For each of them, multiplied by a suitable factor f, there exists another set of two commuting operators, $[fH, Y_{1,2}] = 0$. This can be considered as a generalization of the notion of integrability in quantum mechanics: commuting operators appear not for the Hamiltonian, but for the Hamiltonian multiplied by a factor.

Case I. The first Hamiltonian written in Cartesian corrdinates is given by

$$H_I(x,y;\frac{\omega^2}{2},\frac{A}{2},\frac{B}{2}) = -\frac{1}{2}(\partial_x^2+\partial_y^2)+\frac{\omega^2}{2}(x^2+y^2)+\frac{1}{2}\left(\frac{A}{x^2}+\frac{B}{y^2}\right)\ , \tag{5}$$

where $A, B > -1/8$ are parameters. The corresponding Schroedinger equation separates in three different coordinate systems: Cartesian, polar and

elliptical. The eigenfunctions can be written in the form

$$\Psi_{n,m}(x,y) = x^{p_1} y^{p_2} L_n^{(-1/2+p_1)}(\omega x^2) L_m^{(-1/2+p_2)}(\omega y^2) e^{-\frac{\omega x^2}{2} - \frac{\omega y^2}{2}} \ , \quad (6)$$

where $L_k^{(\alpha)}(z)$ are Laguerre polynomials, $n, m = 0, 1, 2, \ldots$ and the parameters $p_{1,2}$ are defined by $A = p_1(p_1 - 1), B = p_2(p_2 - 1)$. The degree of degeneracy of eigenstates is given by a number of partitions of an integer into the sum of two integers.

We perform a gauge rotation of $H_I(x, y)$, using the ground state eigenfunction $\Psi_{0,0}(x, y)$ as a gauge factor and then a change of variables

$$h^I \equiv \frac{1}{\omega}(\Psi_{0,0}(x,y))^{-1} H_I(x,y) \Psi_{0,0}(x,y)|_{t=\omega x^2, u=\omega y^2} =$$

$$-2t\partial_t^2 - 2u\partial_u^2 + 2t\partial_t + 2u\partial_u - (2p_1+1)\partial_t - (2p_2+1)\partial_u + 1 + p_1 + p_2 \ , \quad (7)$$

with eigenvalues $E(n, m) = n + m$, $n, m = 0, 1, 2, \ldots$.

It is easy to check that after a gauge rotation with the same gauge factor $\Psi_{0,0}(x, y)$ and a change of variables for the integrals of motion, we arrive at the operators

$$\hat{x}_C^I = 2t\partial_t^2 - 2u\partial_u^2 - 2t\partial_t + 2u\partial_u + (2p_1+1)\partial_t - (2p_2+1)\partial_u - p_1 + p_2 \ , \quad (8)$$

$$\hat{x}_R^I \ = 4tu(\partial_t - \partial_u)^2 + 2[(2p_1+1)u - (2p_2+1)t](\partial_t - \partial_u) - (p_1+p_2)^2 \ . \quad (9)$$

These three operators $h^I, \hat{x}_C^I, \hat{x}_R^I$ obey the commutation relations

$$[h^I, \hat{x}_C^I] \ = \ [h^I, \hat{x}_R^I] \ = \ 0 \ ,$$

and

$$[\hat{x}_C^I, \hat{x}_R^I] = 32tu\partial_{tuu}^3 - 32tu\partial_{ttu}^3 - 8t(2p_2+1-2u)\partial_{tt}^2 + 8u(2p_1+1-2t)\partial_{uu}^2 +$$

$$+16[(2p_2+1)t-(2p_1+1)u]\partial_{tu}^2 - 4(2p_1+1)(2p_2+1-2u)\partial_t + 4(2p_2+1)(2p_1+1-2t)\partial_u \quad (10)$$

They generate an infinite-dimensional algebra.

The operators $h^I, \hat{x}_C^I, \hat{x}_R^I$ as well as the commutator $[\hat{x}_C^I, \hat{x}_R^I]$ can be immediately rewritten in terms of the generators $\{J_{1,\ldots,6}\}$ of the maximal parabolic subalgebra of $sl(3, \mathbb{R})$. They have the form

$$h^I \ = \ -2J_3J_1 - 2J_4J_2 + 2J_3 + 2J_4 - (2p_1+1)J_1 - (2p_2+1)J_2 \ , \quad (11)$$

$$\hat{x}_C^I = 2J_3J_1 - 2J_4J_2 - 2J_3 + 2J_4 + (2p_1+1)J_1 - (2p_2+1)J_2 \ , \qquad (12)$$

$$\hat{x}_R^I = 4J_3J_5 + 4J_4J_6 - 8J_3J_4 +$$

$$2(2p_1+1)J_5 - 2(2p_2+1)J_3 - 2(2p_1+1)J_4 + 2(2p_2+1)J_6 \ . \qquad (13)$$

The commutation relation (10) is rewritten as

$$[\hat{x}_C^I, \hat{x}_R^I] = 32J_4J_3(J_2 - J_1) - 16J_4J_6 + 16J_3J_5$$

$$+8(2p_1+1)J_4(J_2 - 2J_1) + 8(2p_2+1)J_3(2J_2 - J_1)$$

$$+ 8(2p_1+1)J_5 - 8(2p_2+1)J_6 + 4(2p_1+1)(2p_2+1)(J_2 - J_1) \ . \qquad (14)$$

Evidently, the operators (11)-(14) preserve a triangular flag of polynomials $\mathcal{P}^{(2)}$ in t, u:

$$h(t) : \mathcal{P}_n(t,u) \mapsto \mathcal{P}_n(t,u)$$

where $\mathcal{P}_n(t,u) = \langle t^p u^q | 0 \le p+q \le n \rangle$. Hence, the operators $h^I, \hat{x}_C^I, \hat{x}_R^I$ are characterized by infinitely many finite-dimensional invariant subspaces and thus possess infinitely many polynomial eigenfunctions.

Case II.

The second superintegrable Hamiltonian can be separated in Cartesian and parabolic coordinates. In Cartesian coordinates it is given by

$$H_{II}(x,y) = -\frac{1}{2}(\partial_x^2 + \partial_y^2) + 2\omega^2 x^2 + \frac{\omega^2}{2}y^2 + \frac{B}{2y^2} \ , \qquad (15)$$

where $B > -1/8$ is a parameter. The eigenfunctions and eigenvalues have the form

$$\Psi_{n,m}(x,y) = y^{p_2} H_n(\sqrt{2\omega}x) L_m^{(-1/2+p_2)}(\omega y^2) e^{-\omega x^2 - \frac{\omega y^2}{2}} \ , \ E_{n,m} = \omega[2(n+m) + p_2 + \frac{3}{2}] \ , \qquad (16)$$

where $n, m = 0, 1, 2, \ldots$; the parameter p_2 is defined by the relation $B = p_2(p_2 - 1)$. The degree of degeneracy is given by the number of partitions of a nonnegative integer into the sum of two nonnegative integers.

We perform a gauge rotation of $H_{II}(x,y)$ with the ground state eigenfunction (16), $\Psi_{0,0}(x,y)$ as a gauge factor and then a change of variables

$$h^{II} \equiv \frac{1}{\omega}(\Psi_{0,0}(x,y))^{-1} H_{II}(x,y) \Psi_{0,0}(x,y)|_{t=\sqrt{2\omega}x, u=\omega y^2} =$$

$$- \partial_t^2 - 2u\partial_u^2 + 2t\partial_t + (2u - 1 - 2p_2)\partial_u + \frac{3}{2} + p_2 \ . \qquad (17)$$

It is easy to check that the operators

$$\hat{x}_C^{II} = 2\partial_t^2 - 4u\partial_u^2 - 4t\partial_t + 2(2u - 1 - 2p_2)\partial_u - 1 + 2p_2 \ , \tag{18}$$

$$\hat{x}_P^{II} \ = -4tu\partial_u^2 + 4u\partial_{tu}^2 - (2u - 1 - 2p_2)\partial_t - 2t(1 + 2p_2)\partial_u \ , \tag{19}$$

generate an infinite dimensional algebra. They obey the commutation relations

$$[h^{II}, \hat{x}_C^{II}] \ = \ [h^{II}, \hat{x}_P^{II}] \ = \ 0 \ ,$$

and

$$[\hat{x}_C^{II}, \hat{x}_P^{II}] = -32u\partial_{tuu}^3 - 16(1 + 2p_2 - 2u)\partial_{tu}^2 + 32tu\partial_{uu}^2 +$$

$$+ 8(1 + 2p_2 - 2u)\partial_t + 16t(1 + 2p_2)\partial_u] \ . \tag{20}$$

The operators $h^{II}, \hat{x}_C^{II}, \hat{x}_P^{II}$ as well as the commutator $[\hat{x}_C^{II}, \hat{x}_P^{II}]$ can be immediately rewritten in terms of the generators (4) and have the form

$$h^{II} = -J_1J_1 - 2J_4J_2 + 2J_3 + 2J_4 - (1 + 2p_2)J_2 + \frac{3}{2} + p_2 \ , \tag{21}$$

$$\hat{x}_C^{II} = 2J_1J_1 - 4J_4J_2 - 4J_3 + 4J_4 - 2(1 + 2p_2)J_2 - 1 + 2p_2 \ , \tag{22}$$

$$\hat{x}_P^{II} \ = \ -4J_4J_6 + 4J_1J_4 - 2J_5 + (1 + 2p_2)J_1 - 2(1 + 2p_2)J_6 \ , \tag{23}$$

Evidently, the operators (21),(22),(23) preserve the same triangular flag of polynomials as in the Case I but in variables x, u:

$$h(x, u) : \mathcal{P}_n(x, u) \mapsto \mathcal{P}_n(x, u)$$

where $\mathcal{P}_n(x, u) = \langle x^p u^q | 0 \leq p + q \leq n \rangle$. Thus, the operators $h^{II}, \hat{x}_C^{II}, \hat{x}_P^{II}$ have infinitely many finite dimensional invariant subspaces and infinitely many polynomial eigenfunctions.

Case III.

The third superintegrable Hamiltonian

$$H_{III}(x, y) = -\frac{1}{2}(\partial_x^2 + \partial_y^2) + \frac{\alpha}{2r} + \frac{1}{4r^2}\left(\frac{\beta_1}{\cos^2\frac{\phi}{2}} + \frac{\beta_2}{\sin^2\frac{\phi}{2}}\right) \ , \tag{24}$$

where $\beta_{1,2} > -1/8$ are parameters and $x = r\cos\phi, y = r\sin\phi$. It admits the separation of variables in polar and parabolic coordinates. In parabolic coordinates it has the form

$$H_{III}(\xi,\eta) = -\frac{1}{2}\frac{1}{\xi^2+\eta^2}(\partial_\xi^2+\partial_\eta^2) + \frac{1}{\xi^2+\eta^2}\left(2\alpha + \frac{\beta_1}{\xi^2} + \frac{\beta_2}{\eta^2}\right) , \tag{25}$$

with $x = \frac{1}{2}(\xi^2 - \eta^2)$, $y = \xi\eta$. The eigenfunctions corresponding to the energy E are

$$\Psi_{n,m} = \xi^{p_1}\eta^{p_2} L_n^{(-1/2+p_1)}(\sqrt{-2E}\xi^2) L_m^{(-1/2+p_2)}(\sqrt{-2E}\eta^2) e^{-\sqrt{-E/2}(\xi^2+\eta^2)} \tag{26}$$

where $2\beta_1 = p_1(p_1-1), 2\beta_2 = p_2(p_2-1)$. It is easy to see that the Schroedinger equation $H_{III}\Psi = E\Psi$ can be transformed into

$$[-\frac{1}{2}(\partial_\xi^2+\partial_\eta^2) - E(\xi^2+\eta^2) + \frac{\beta_1}{\xi^2} + \frac{\beta_2}{\eta^2}]\Psi = -2\alpha\Psi . \tag{27}$$

We introduce the notation

$$Q_{III} \equiv (\xi^2+\eta^2)(H_{III} - E) - 2\alpha . \tag{28}$$

Equation (27) can be written as

$$Q_{III}\Psi = -2\alpha\Psi , \tag{29}$$

and Q^{III} can be related to H^I

$$Q_{III} = H_I(\xi,\eta; -E, \beta_1, \beta_2) , \tag{30}$$

(cf. (5)). We draw the striking conclusion that the Hamiltonian of the first problem (Case I) written in *Cartesian* coordinates coincides with a modified third Hamiltonian Q^{III} written in *parabolic* coordinates (!). The parameter (-2α) plays the role of spectral parameter which is the energy in Case I, $(-2\alpha) \longleftrightarrow E^I$. Thus, the analysis performed for the Case I can be repeated for this case.

We perform a gauge rotation of the operator Q_{III} with a gauge factor given by the multiplier figuring in eq.(26),

$$M_{III} = \xi^{p_1}\eta^{p_2} e^{-\sqrt{-E/2}(\xi^2+\eta^2)} . \tag{31}$$

Notice that in this case, contrary to those of H_I and H_{II}, the gauge factor is not universal. It depends on the energy E. Thus E in the multiplier M is the considered energy, not the ground state one.

Thus we have

$$q^{III} \equiv \frac{1}{\sqrt{-2E}} M_{III}^{-1} Q_{III}(x,y) M_{III} \ ,$$

and with a change of coordinates $t = \sqrt{-2E}\xi^2, u = \sqrt{-2E}\eta^2$ we get

$$q^{III} = -2t\partial_t^2 - 2u\partial_u^2 + 2t\partial_t + 2u\partial_u - (2p_1+1)\partial_t - (2p_2+1)\partial_u + 1 + p_1 + p_2 \ . \quad (32)$$

This operator coincides *exactly* with the operator h^I (7). Its spectrum is equal to $-2\alpha/\sqrt{-E} = 2(n+m) + 1 + p_1 + p_2 \ , \ n, m = 0, 1, 2, \ldots$.

Thus, the operator q^{III} commutes with $\hat{x}_C^I, \hat{x}_R^I$. We shall call these operators $\hat{x}_C^{III}, \hat{x}_R^{III}$. The operator q^{III} can be rewritten in terms of the generators $\{J_{1,\ldots,6}\}$ of the maximal parabolic subalgebra of $sl(3,\mathbb{R})$ (see (4)). We see that the Hamiltonian (24) is exactly solvable.

The above observation gives rise to an interesting question about the connection between the operators commuting with original Hamiltonian (24) and the operators $\hat{x}_C^{III}, \hat{x}_R^{III}$.

The operators that commute with Q_{III} of eq. (28) can be read off from those of case H_I. They are in parabolic coordinates

$$X_C^{III} = -\frac{1}{2}(\partial_\xi^2 - \partial_\eta^2) - E(\xi^2 - \eta^2) + \frac{\beta_1}{\xi^2} - \frac{\beta_2}{\eta^2},$$

$$X_R^{III} = (\xi\partial_\eta - \eta\partial_\xi)^2 - 2(\xi^2+\eta^2)(\frac{\beta_1}{\xi^2} + \frac{\beta_2}{\eta^2}). \quad (33)$$

The operators that commute with the original Hamiltonian H_{III} of eq. (25), if written in parabolic coordinates, are [3, 4, 17]

$$X_P = \frac{1}{\xi^2+\eta^2}(\eta^2\partial_\xi^2 - \xi^2\partial_\eta^2 + 2\alpha(\xi^2-\eta^2) - 2\beta_1\frac{\eta^2}{\xi^2} + 2\beta_2\frac{\xi^2}{\eta^2}),$$

$$X_R = (\xi\partial_\eta - \eta\partial_\xi)^2 - 2(\xi^2+\eta^2)(\frac{\beta_1}{\xi^2} + \frac{\beta_2}{\eta^2}). \quad (34)$$

Let us consider, quite generally a Hamiltonian H and an operator Q, defined by the relation

$$H = \frac{Q+K}{\xi^2+\eta^2} + E, \quad (35)$$

where K is a constant and E is the energy. Let X be an operator commuting with the Hamiltonian: $[H, X] = 0$. The operator Q then satisfies

$$[Q, X] = \left(\xi^2+\eta^2\right)\left[X, \frac{1}{\xi^2+\eta^2}\right]\left(\xi^2+\eta^2\right)(H-E)\,. \quad (36)$$

Thus, if the operator X commutes with H (strongly, as an operator), it commutes weakly, on functions Ψ satisfying $(H-E)\,\Psi=0$, with Q. So, in order to relate operators that commute with H and those that commute with Q, we have to consider linear combinations of the type $AX_P+BX_R+f\,(\xi,\eta)$ $(H-E)$. Here A and B are constants, but $f\,(\xi,\eta)$ can be any function, since $H-E$ vanishes on the *"energy shell"*.

Let us return to the problem at hand, i.e. the system characterized by the Hamiltonian H_{III}, or equivalently, by the operator Q_{III}. We have the following simple relation between the original integrals X_P and X_R and the modified ones (33)

$$X_C^{III}=-X_P+\left(\xi^2-\eta^2\right)(H-E)\,,\qquad X_R^{III}=X_R. \tag{37}$$

Case IV.

The fourth superintegrable Hamiltonian admits the separation of variables in two mutually perpendicular parabolic systems of coordinates. In the usual parabolic coordinates ξ,η it has the form

$$H_{IV}(\xi,\eta)=-\frac{1}{2}\frac{1}{\xi^2+\eta^2}(\partial_\xi^2+\partial_\eta^2)+\frac{1}{\xi^2+\eta^2}\left(2\alpha+\beta\xi+\gamma\eta\right)\,. \tag{38}$$

The eigenfunctions are products of Laguerre polynomials times an energy dependent multiplier, namely

$$M_{IV}=(\xi-\beta/2E)^{p_1}(\eta-\gamma/2E)^{p_2}e^{-\sqrt{-E/2}[(\xi-\beta/2E)^2+(\eta-\gamma/2E)^2]}$$

with a condition $\alpha+\frac{\beta^2}{4E}+\frac{\gamma^2}{4E}+\sqrt{-2E}=0$ and $p_1,p_2=0,1$.

The corresponding Schroedinger equation $H_{IV}\Psi=E\Psi$ can be rewritten as

$$[-\frac{1}{2}(\partial_\xi^2+\partial_\eta^2)-E(\xi-\frac{\beta}{2E})^2-E(\eta-\frac{\gamma}{2E})^2]\Psi=(-2\alpha-\frac{\beta^2+\gamma^2}{4E})\Psi\ . \tag{39}$$

For the operator on the left hand side we introduce the notation

$$Q_{IV}\equiv(\xi^2+\eta^2)(H^{IV}-E)-2\alpha-\frac{\beta^2+\gamma^2}{4E}\ .$$

Equation (39) can be written as

$$Q_{IV}\Psi=\tilde{\alpha}\Psi\ ,$$

with the new spectral parameter $\tilde{\alpha}=-2\alpha-\frac{\beta^2+\gamma^2}{4E}$.

The operator Q^{IV} can be related to H^I

$$Q_{IV} = H_I(\xi, \eta; -E, 0, 0) \ , \tag{40}$$

(cf. (5), (30)). We see that similarly to the Case III the Hamiltonian of Case I written in *Cartesian* coordinates coincides with a modified fourth Hamiltonian Q_{IV} written in *parabolic* coordinates. The parameter $\tilde{\alpha}$ plays the role of a spectral parameter which was the energy in the first case, $\tilde{\alpha} \longleftrightarrow E^I$. Thus, the analysis performed for the Case I can be again repeated for this case.

Writing the gauge rotated operator Q_{IV} in new coordinates

$$t = \sqrt{-2E}(\xi - \beta/2E)^2, u = \sqrt{-2E}(\eta - \gamma/2E)^2$$

we get

$$q^{IV} \equiv \frac{1}{\sqrt{-2E}} M_{IV}^{-1} Q_{IV}(x, y) M_{IV}$$

$$= -2t\partial_t^2 - 2u\partial_u^2 + 2t\partial_t + 2u\partial_u - (2p_1 + 1)\partial_t - (2p_2 + 1)\partial_u + 1 + p_1 + p_2 \ ,$$

This operator coincides *exactly* with the operator h^I (7) and q^{III} (32). Its spectrum is equal to $2(n + m) + 1 + p_1 + p_2 \ , \ n, m = 0, 1, 2, \ldots$.

Thus, the operator q^{IV} commutes with $\hat{x}_C^I, \hat{x}_R^I$. We shall call these operators $\hat{x}_C^{IV}, \hat{x}_R^{IV}$. The operator q^{IV} (as well as $\hat{x}_C^{IV}, \hat{x}_R^{IV}$) can be rewritten in terms of the generators $\{J_{1,\ldots,6}\}$ of the maximal parabolic subalgebra of $sl(3, \mathbb{R})$ (see (4)) and hence is exactly solvable.

As in the Case III one can give the connection between the operators commuting with original Hamiltonian (38) and the operators $\hat{x}_C^{IV}, \hat{x}_R^{IV}$. The operators commuting with H_{IV} are [3, 4, 17]

$$X_1 = \frac{1}{2\left(\xi^2 + \eta^2\right)} \left\{\eta^2 \partial_\xi^2 - \xi^2 \partial_\eta^2 + 2\alpha\left(\xi^2 - \eta^2\right) + 2\xi\eta\left(\gamma\xi - \beta\eta\right)\right\} \ ,$$

$$X_2 = \frac{1}{\left(\xi^2 + \eta^2\right)} \left\{\xi\eta\left(\partial_\xi^2 + \partial_\eta^2\right) + \left(-\beta\eta + \gamma\xi\right)\left(\xi^2 - \eta^2\right) - 4\alpha\xi\eta\right\} - \partial_{\xi\eta}^2 \ . \tag{41}$$

Two *second order* differential operators commuting with Q_{IV} of eq. (40) can be related to (41) and are given by

$$Q^{(1)} = -2X_1 + \left(\xi^2 - \eta^2\right)\left(H_{IV} - E\right) + \frac{\gamma^2 - \beta^2}{4E} \ , \tag{42}$$

$$Q^{(2)} = X_2 + 2\xi\eta\left(H_{IV} - E\right) - \frac{\beta\gamma}{2E} \ . \tag{43}$$

From this point of view the fourth superintegrable system is particularly simple. Explicitly, we have a new “Hamiltonian”

$$Q_{IV} = -\frac{1}{2}\left(\partial_\xi^2 + \partial_\eta^2\right) - E\left[\left(\xi - \frac{\beta}{2E}\right)^2 + \left(\eta - \frac{\gamma}{2E}\right)^2\right] , \qquad (44)$$

which corresponds to the harmonic oscillator, while one of the commuting operators satisfies

$$Q^{(1)} = X_C^I , \qquad (45)$$

where the integral X_C^I (see (8)) is written in the coordinates $(\xi - \beta/2E), (\eta - \gamma/2E)$.

In turn, the operator $\widehat{X}_R^I$ reduces to

$$\widehat{X}_R^I = [(\xi - \frac{\beta}{2E})\partial_\eta - (\eta - \frac{\gamma}{2E})\partial_\xi]^2 \equiv L_z^2 . \qquad (46)$$

Thus, the superintegrable system characterized by the Hamiltonian $H_{IV}(\xi, \eta)$ has been reduced to a harmonic oscillator with ”frequency” $\omega = \sqrt{-E/2}$ and a displaced equilibrium point $\xi = \beta/2E$, $\eta = \gamma/2E$. It is well-known that the harmonic oscillator is invariant under an $SU(2)$ group. Indeed, we find that the ”Hamiltonian” Q_{IV} commutes with $Q^{(1,2)}$ and L_z. The operators (42), (43), (44) and L_z form the basis of a $u(2)$ symmetry algebra with Q_{IV} as its center.

3 Conclusions

In general, integrability of a quantum system does not guarantee that spectrum and eigenfunctions can be found in sufficiently explicit form. The simplest example of this situation is given by one-dimensional quantum dynamics which is integrable for any potential. The main message of the present work is that the superintegrable systems on E_2 with the integrals given by second order differential operators are exactly solvable as well. We conjecture that the property of exact solvability will remain valid for higher dimensional superintegrable systems of the above mentioned type.

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